

# A Unified Treatment of the Characters of $SU(2)$ and $SU(1,1)$

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## Abstract

The character problems of  $SU(2)$  and  $SU(1,1)$  are reexamined from the standpoint of a physicist by employing the Hilbert space method which is shown to yield a completely unified treatment for  $SU(2)$  and the discrete series of representations of  $SU(1,1)$ . For both the groups the problem is reduced to the evaluation of an integral which is invariant under rotation for  $SU(2)$  and Lorentz transformation for  $SU(1,1)$ . The integrals are accordingly evaluated by applying a rotation to a unit position vector in  $SU(2)$  and a Lorentz transformation to a unit  $SO(2,1)$  vector which is time-like for the elliptic elements and space-like for the hyperbolic elements in  $SU(1,1)$ . The details of the procedure for the principal series of representations of  $SU(1,1)$  differ substantially from those of the discrete series.

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## I. INTRODUCTION

A major tool in group representation theory is the theory of character. The importance of the concept of character of a representation stems from the fact that for a semisimple Lie group every unitary irreducible representation is uniquely determined by its character. The simplification effected by such an emphasis is obvious; in particular the formal processes of direct sum and direct product as applied to representations are reflected in ordinary sum and multiplication of characters.

For finite dimensional representations character is traditionally defined as the sum of the eigenvalues of the representation matrix. It should be pointed out that the unitary operators of an infinite dimensional Hilbert space do not have character in this sense since the infinite sum consists of numbers of unit modulus. For example for  $g = e$  one has  $D(e) = I$  and the sum of the diagonal elements of the infinite dimensional unit matrix is  $\infty$ . We now briefly give the Gel'fand-Naimark<sup>1</sup> definition of character which introduces the concept through the group ring as a generalized function on the group manifold.

We denote by  $X$  the set of infinitely differentiable functions  $x(g)$  on the group, which are equal to zero outside a bounded set. If  $g \rightarrow T_g$  be a representation of the group  $G$  we set

$$T_x = \int x(g) T_g d\mu(g) \tag{1}$$

where  $d\mu(g)$  is the left and right invariant measure (assumed coincident) on  $G$  and the integration extends over the entire group manifold.

The product  $T_{x_1} T_{x_2}$  can be written in the form,

$$T_{x_1} T_{x_2} = \int x(g) T_g d\mu(g)$$

where

$$x(g) = \int x_1(g_1)x_2(g_1^{-1}g)d\mu(g_1) \quad (2)$$

The function  $x(g)$  defined by equation ( 2) will be called the product of the functions  $x_1, x_2$  and denoted by  $x_1x_2(g)$ .

Let us suppose that  $g \rightarrow T_g$  is a unitary representation of the group  $G$  realized in the Hilbert space  $H$  of the functions  $f(z)$  with the scalar product

$$(f, g) = \int \overline{f(z)}g(z)d\lambda(z)$$

where  $d\lambda(z)$  is the measure in  $H$ .

Then the operator  $T_x$  is an integral operator with a kernel:

$$T_x f(z) = \int K(z, z_1)f(z_1)d\lambda(z_1)$$

It then follows that  $K(z, z_1)$  is a positive definite Hilbert-Schmidt kernel, satisfying

$$\int |K(z, z_1)|^2 d\lambda(z)d\lambda(z_1) < \infty$$

Such a kernel has a trace  $Tr(T_x)$  where

$$Tr(T_x) = \int K(z, z)d\lambda(z)$$

Using the definition in equation ( 1) one can prove that  $Tr(T_x)$  can be written in the form

$$Tr(T_x) = \int x(g)\pi(g)d\mu(g) \quad (3)$$

The function  $\pi(g)$  is the character of the representation  $g \rightarrow T_g$ . It should be noted that in this definition the matrix representation of the group does not appear and, as will be shown below, it makes a complete synthesis of the finite and infinite dimensional irreducible unitary representations.

The character of the complex and real unimodular groups was evaluated by Gel'fand and coworkers<sup>2,3</sup>. The real group  $SL(2, \mathbb{R})$  turns out to be more involved than the complex

group particularly because of the presence of the discrete series of unitary irreducible representations (unirreps). The main problem in the Gel'fand-Naimark theory of character is the construction of the integral kernel  $K(z, z_1)$  which requires a judicious choice of the carrier space of the representation. The representations of the positive discrete series  $D_k^+$  were realised by Gel'fand and coworkers<sup>3</sup> in the space of the functions on the half-line  $R^+$  and those of the negative discrete series  $D_k^-$  on the half-line  $R^-$ . The integral kernel of the group ring was determined by them essentially for the reducible representation  $D_k^+ \oplus D_k^-$  which considerably complicates the subsequent computation of the character of a single irreducible representation. It is the object of this paper to re-examine the character problem of  $SU(1,1)$  (or  $SL(2,R)$ ) from a physicist's standpoint by using the Hilbert space method developed by Bargmann<sup>4</sup> and Segal<sup>5</sup> in which computations can be carried out within a single unirrep of the positive or negative discrete series. This method not only simplifies the crucial problem of construction of the integral kernel of the group ring but serves as the key to the synthesis of the finite and infinite dimensional representations mentioned above. The Hilbert space method as applied to the evaluation of the characters of  $SU(2)$  and  $SU(1,1)$  proceeds along entirely parallel lines. For  $SU(2)$  the problem essentially reduces to the evaluation of an integral of the form

$$\int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi \left( \cos \frac{\theta_0}{2} - i \sin \frac{\theta_0}{2} \hat{n} \cdot \hat{r} \right)^{2j} \quad (4)$$

where  $\hat{n}$  and  $\hat{r}$  are unit position vectors,  $\hat{n}$  being fixed and  $\hat{r}(\theta, \phi)$  being the variable of integration. This integral is easily evaluated by rotating the co-ordinate axes such that the 3-axis (Z-axis) coincides with  $\hat{n}$

The character of the elliptic elements of  $SU(1,1)$  is given by an integral closely resembling the above:

$$\int_{\tau=0}^{\infty} d\tau \sinh \tau \int_0^{2\pi} d\phi \left[ \cos \frac{\theta_0}{2} - i \sin \frac{\theta_0}{2} \hat{n} \cdot \hat{r} \right]^{-2k} \quad (5)$$

where  $\hat{n}$  and  $\hat{r}$  are a pair of unit time-like  $\text{SO}(2,1)$  vectors,  $\hat{n}$ , as before, being fixed and  $\hat{r}$  the variable of integration. This integral is evaluated by an appropriate Lorentz transformation such that the time axis points along the fixed time-like  $\text{SO}(2,1)$  vector  $\hat{n}$ . For the hyperbolic elements of  $\text{SU}(1,1)$  the above integral is replaced by

$$\int_0^\infty d\tau \sinh \tau \int d\phi \left[ \epsilon \cosh \frac{\sigma}{2} - i \sinh \frac{\sigma}{2} \hat{n} \cdot \hat{r} \right]^{-2k}$$

where  $\epsilon = \pm 1$ ,  $\hat{n}$  is a unit space-like and  $\hat{r}$  is a unit time-like  $\text{SO}(2,1)$  vector. This integral is once again evaluated by a Lorentz transformation such that the first space axis (X-axis) coincides with the fixed space-like vector  $\hat{n}$ . The explicit evaluation is, however, a little lengthier than that for the elliptic elements.

For the principal series of representations the carrier space is chosen to be the traditional Hilbert space<sup>6</sup> of functions defined on the unit circle. Although the broad outlines of the procedure is the same as above the details differ substantially from those of the discrete series. An important feature of the principal series of representations is that the elliptic elements of  $\text{SU}(1,1)$  do not contribute to its character.

## II. THE GROUP $\text{SU}(2)$ AND THE DISCRETE SERIES OF REPRESENTATIONS OF $\text{SU}(1,1)$

To make this paper self contained we describe the basic properties of the Hilbert spaces of analytic functions for  $\text{SU}(2)$  and  $\text{SU}(1,1)$ .

### A. The Group $\text{SU}(2)$

The group  $\text{SU}(2)$  consists of  $2 \times 2$  unitary, unimodular matrices

$$u = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1 \quad (6)$$

We know that every unitary, unimodular matrix  $u$  can be diagonalized by a unitary unimodular matrix  $v$  so that

$$u = v\epsilon(\theta_0)v^{-1} \quad (7)$$

where  $\epsilon(\theta_0)$  is the diagonal form of  $u$  :

$$\epsilon(\theta_0) = \begin{pmatrix} e^{\frac{i\theta_0}{2}} & 0 \\ 0 & e^{-\frac{i\theta_0}{2}} \end{pmatrix}$$

Since  $v$  is also an  $SU(2)$  matrix it can be factorized in terms of Euler angles as,

$$v = \epsilon(\eta)a(\tau)\epsilon(\chi), \quad (8)$$

where

$$a(\tau) = \begin{pmatrix} \cos \frac{\tau}{2} & \sin \frac{\tau}{2} \\ -\sin \frac{\tau}{2} & \cos \frac{\tau}{2} \end{pmatrix}$$

We therefore obtain the following parametrization of  $u$  :

$$u = \epsilon(\eta)a(\tau)\epsilon(\theta_0)a^{-1}(\tau)\epsilon^{-1}(\eta). \quad (9)$$

The parametrization ( 9) yields

$$\alpha = \cos \frac{\theta_0}{2} + i \sin \frac{\theta_0}{2} \cos \tau \quad (10)$$

$$\beta = -ie^{i\eta} \sin \frac{\theta_0}{2} \sin \tau \quad (11)$$

The representations of  $SU(2)$  will be realized in the Bargmann-Segal space  $B(C_2)$  which consists of entire analytic functions  $\phi(z_1, z_2)$  where  $z_1$  and  $z_2$  are spinors transforming according to the fundamental representation of  $SU(2)$  :

$$(z'_1, z'_2) = (z_1, z_2)u \quad (12)$$

The action of the finite element of the group in  $B(C_2)$  is given by,

$$T_u \phi(z_1, z_2) = \phi(\alpha z_1 - \bar{\beta} z_2, \beta z_1 + \bar{\alpha} z_2)$$

To decompose  $B(C_2)$  into the direct sum of the subspaces  $B_j(C)$  invariant under  $SU(2)$  we introduce Schwinger's angular momentum operators in  $B(C_2)$

$$\begin{aligned} J_1 &= \frac{1}{2} \left( z_1 \frac{\partial}{\partial z_2} + z_2 \frac{\partial}{\partial z_1} \right) \\ J_2 &= -\frac{i}{2} \left( z_1 \frac{\partial}{\partial z_2} - z_2 \frac{\partial}{\partial z_1} \right) \\ J_3 &= \frac{1}{2} \left( z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2} \right) \end{aligned}$$

Explicit calculation yields

$$\vec{J}^2 = J_1^2 + J_2^2 + J_3^2 = \frac{K}{2} \left( \frac{K}{2} + 1 \right)$$

where  $K$  stands for the operator

$$K = \left( z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} \right)$$

Since  $K$  commutes with all the components of the angular momentum in an irreducible representation it can be replaced by  $2j$ ,  $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$ . The subspace  $B_j(C)$  is, therefore, the space of homogeneous polynomials of degree  $2j$  in  $z_1, z_2$  :

$$\phi(z_1, z_2) = [(2j)!]^{-\frac{1}{2}} z_2^{2j} f(z), z = \frac{z_1}{z_2} \quad (13)$$

where the numerical factor  $[(2j)!]^{-\frac{1}{2}}$  is introduced for convenience. If we restrict ourselves to functions of the form ( 13) the finite element of the group is given by

$$T_u f(z) = (\beta z + \bar{\alpha})^{2j} f \left( \frac{\alpha z - \bar{\beta}}{\beta z + \bar{\alpha}} \right) \quad (14)$$

This representation is unitary with respect to the scalar product

$$(f, g) = \int \overline{f(z)} g(z) d\lambda(z) \quad (15)$$

where

$$d\lambda(z) = \frac{(2j+1)}{\pi} (1 + |z|^2)^{-2j-2} d^2z, \quad (16)$$

$$d^2z = dx dy, z = x + iy$$

The principal vector in this space is given by

$$e_z(z_1) = (1 + \bar{z}z_1)^{2j} \quad (17)$$

so that

$$f(z) = \int (1 + z\bar{z}_1)^{2j} f(z_1) d\lambda(z_1) \quad (18)$$

We now construct the group ring which consists of the operators

$$T_x = \int x(u) T_u d\mu(u),$$

where  $d\mu(u)$  is the invariant measure on  $SU(2)$  and  $x(u)$  is an arbitrary test function on the group, which vanishes outside a bounded set. The action of the group ring is then given by

$$T_x f(z) = \int x(u) (\beta z + \bar{\alpha})^{2j} f\left(\frac{\alpha z - \bar{\beta}}{\beta z + \bar{\alpha}}\right) d\mu(u)$$

We now use the reproducing kernel as given by ( 18) to write

$$f\left(\frac{\alpha z - \bar{\beta}}{\beta z + \bar{\alpha}}\right) = \int \left[1 + \frac{(\alpha z - \bar{\beta})\bar{z}_1}{(\beta z + \bar{\alpha})}\right]^{2j} f(z_1) d\lambda(z_1)$$

Thus

$$T_x f(z) = \int K(z, z_1) f(z_1) d\lambda(z_1)$$

where the kernel  $K(z, z_1)$  is given by,



$$K(z, z_1) = \int x(u)(\beta z + \bar{\alpha})^{2j} \left[ 1 + \frac{(\alpha z - \bar{\beta})\bar{z}_1}{(\beta z + \bar{\alpha})} \right]^{2j} d\mu(u) \quad (19)$$

Since the kernel  $K(z, z_1)$  is of the Hilbert-Schmidt type we have

$$Tr(T_x) = \int K(z, z) d\lambda(z)$$

Using the definition ( 19) of the kernel we have

$$Tr(T_x) = \int x(u)\pi(u)d\mu(u)$$

where

$$\pi(u) = \int (\beta z + \bar{\alpha})^{2j} \left[ 1 + \frac{(\alpha z - \bar{\beta})\bar{z}}{(\beta z + \bar{\alpha})} \right]^{2j} d\lambda(z) \quad (20)$$

Setting

$$z = \tan \frac{\theta}{2} e^{i\phi}, \quad 0 \leq \theta < \pi, 0 \leq \phi \leq 2\pi$$

and using the parametrization ( 10,11) we obtain after some calculations,

$$\pi(u) = \frac{2j+1}{4\pi} \int_{\theta=0}^{\pi} d\theta \sin \theta \int_0^{2\pi} d\phi \left[ \cos \frac{\theta_0}{2} - i \sin \frac{\theta_0}{2} (\cos \tau \cos \theta + \sin \tau \sin \theta \cos \phi) \right]^{2j} \quad (21)$$

If we now introduce the unit vectors  $\hat{n}$  and  $\hat{r}$  as

$$\hat{n} = (\sin \tau, 0, \cos \tau), \hat{r} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

the equation ( 21) can be written as

$$\pi(u) = \frac{2j+1}{4\pi} \int_{\theta=0}^{\pi} d\theta \sin \theta \int_0^{2\pi} d\phi \left[ \cos \frac{\theta_0}{2} - i \sin \frac{\theta_0}{2} \hat{n} \cdot \hat{r} \right]^{2j} \quad (22)$$

We now rotate the coordinate system such that the 3-axis (Z-axis) coincides with the fixed vector  $\hat{n}$ . Thus

$$\pi(u) = \frac{2j+1}{2} \int_{\theta=0}^{\pi} \left[ \cos \frac{\theta_0}{2} - i \sin \frac{\theta_0}{2} \cos \theta \right]^{2j} \sin \theta d\theta$$

The above integral is quite elementary and yields

$$\pi(u) = \frac{\sin(j + \frac{1}{2})\theta_0}{\sin \frac{\theta_0}{2}}$$

## B. The Group SU(1,1)

The group SU(1,1) consists of pseudo-unitary, unimodular matrices

$$u = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad \det(u) = |\alpha|^2 - |\beta|^2 = 1 \quad (23)$$

and is isomorphic to the group SL(2,R) of real unimodular matrices,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \quad \det(g) = ad - bc = 1 \quad (24)$$

A particular choice of the isomorphism kernel is

$$\eta = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \quad (25)$$

so that

$$u = \eta g \eta^{-1}$$

$$\alpha = \frac{1}{2}[(a+d) - i(b-c)]; \quad \beta = \frac{1}{2}[(b+c) - i(a-d)] \quad (26)$$

The elements of the group SU(1,1) may be divided into three subsets : a) elliptic, b) hyperbolic and c) parabolic. We define them as follows. Let  $\alpha = \alpha_1 + i\alpha_2$  and  $\beta = \beta_1 + i\beta_2$  so that

$$\alpha_1^2 + \alpha_2^2 - \beta_1^2 - \beta_2^2 = 1$$

The elliptic elements are those for which

$$\alpha_2^2 - \beta_1^2 - \beta_2^2 > 0$$

Hence if we set

$$\alpha'_2 = \sqrt{\alpha_2^2 - \beta_1^2 - \beta_2^2}$$

we have

$$\alpha_1^2 + \alpha_2'^2 = 1$$

so that  $-1 < \alpha_1 < 1$ .

On the other hand the hyperbolic elements of  $SU(1,1)$  are those for which

$$\alpha_2^2 - \beta_1^2 - \beta_2^2 < 0$$

Hence if we write

$$\alpha'_2 = \sqrt{\beta_1^2 + \beta_2^2 - \alpha_2^2}$$

we have

$$\alpha_1^2 - \alpha_2'^2 = 1 \tag{27}$$

so that  $|\alpha_1| > 1$ .

We exclude the parabolic class corresponding to

$$\alpha_2 = \sqrt{\beta_1^2 + \beta_2^2}$$

as this is a submanifold of lower dimensions.

If we diagonalize the  $SU(1,1)$  matrix ( 23), the eigenvalues are given by

$$\lambda = \alpha_1 \pm \sqrt{\alpha_1^2 - 1}$$

We shall consider the elliptic case  $-1 < \alpha_1 < 1$  first. Thus, setting  $\alpha_1 = \cos(\frac{\theta_0}{2})$ ,  $0 < \theta_0 < 2\pi$  we have  $\lambda = \exp(\pm i \frac{\theta_0}{2})$ . We shall now show that every elliptic element of  $SU(1,1)$  can be diagonalized by a pseudounitary transformation *ie*

$$v^{-1}uv = \epsilon(\theta_0), \quad \epsilon(\theta_0) = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}, \quad \delta_1 = \bar{\delta}_2 = e^{i\frac{\theta_0}{2}}, \quad (28)$$

where  $v \in \text{SU}(1,1)$

To prove this we first note that equation ( 28) can be written as

$$uv_1 = \delta_1 v_1, uv_2 = \delta_2 v_2 \quad (29)$$

where

$$v_1 = \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix}, v_2 = \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix}$$

Thus  $v_1$  and  $v_2$  are the eigenvectors of the matrix  $u$  belonging to the eigenvalues  $\delta_1$  and  $\delta_2$  respectively. Hence  $v_1$  and  $v_2$  are linearly independent so that  $\det(v) \neq 0$ . We normalize the matrix  $v$  such that

$$\det(v) = v_{11}v_{22} - v_{12}v_{21} = 1$$

We now show that the eigenvectors  $v_1$  and  $v_2$  are pseudoorthogonal *ie* orthogonal with respect to the metric

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In fact from the equation ( 29) we easily obtain

$$\delta_1^2 v_2^\dagger \sigma_3 v_1 = v_2^\dagger u^\dagger \sigma_3 u v_1$$

Using the pseudounitariness of the matrix  $u \in \text{SU}(1,1)$  we immediately obtain

$$(v_2^\dagger \sigma_3 v_1) = 0$$

If we further normalize

$$v_1^\dagger \sigma_3 v_1 = 1,$$

we easily deduce

$$v_{21} = \bar{v}_{12}, v_{22} = \bar{v}_{11}$$

Thus for the elliptic elements of  $SU(1,1)$  the transformation matrix  $v$  defined by ( 28) is also an  $SU(1,1)$  matrix. Since every matrix  $v \in SU(1,1)$  can be written as

$$v = \epsilon(\eta)a(\sigma)\epsilon(\theta),$$

where

$$a(\sigma) = \begin{pmatrix} \cosh \frac{\sigma}{2} & \sinh \frac{\sigma}{2} \\ \sinh \frac{\sigma}{2} & \cosh \frac{\sigma}{2} \end{pmatrix} \quad (30)$$

we immediately obtain

$$u = \epsilon(\eta)a(\sigma)\epsilon(\theta_0)a^{-1}(\sigma)\epsilon^{-1}(\eta)$$

The above parametrization yields

$$\alpha = \cos \frac{\theta_0}{2} + i \sin \frac{\theta_0}{2} \cosh \sigma \quad (31)$$

$$\beta = -ie^{i\eta} \sin \frac{\theta_0}{2} \sinh \sigma \quad (32)$$

We now consider the hyperbolic elements of  $SU(1,1)$  satisfying equation ( 27). Since now  $|\alpha_1| > 1$ , setting  $\alpha_1 = \epsilon \cosh \frac{\sigma}{2}$ ,  $\epsilon = \text{sgn } \lambda$ , we obtain the eigenvalues as  $\epsilon e^{\pm \epsilon \frac{\sigma}{2}}$ . Since the diagonal matrix

$$\epsilon(\sigma) = \begin{pmatrix} \text{sgn} \lambda e^{\text{sgn} \lambda (\frac{\sigma}{2})} & 0 \\ 0 & \text{sgn} \lambda e^{-\text{sgn} \lambda (\frac{\sigma}{2})} \end{pmatrix} \quad (33)$$

belongs to  $SL(2, \mathbb{R})$ , it can be regarded as the diagonal form of the matrix  $g$  given by equations ( 24) and ( 26) with  $|\alpha_1| = \frac{|a+d|}{2} > 1$ . Henceforth we shall take  $\text{sgn} \lambda = 1$ . The other case  $\text{sgn} \lambda = -1$  can be developed in an identical manner.

An analysis parallel to the one for the elliptic elements shows that for  $(a + d) > 2$  every matrix  $g \in \text{SL}(2, \mathbb{R})$  can be diagonalized also by a matrix  $v \in \text{SL}(2, \mathbb{R})$ . Thus

$$g = v\epsilon(\sigma)v^{-1}$$

Since every matrix  $v \in \text{SL}(2, \mathbb{R})$  can be decomposed as

$$v = e(\theta)a(\rho)\epsilon(\alpha),$$

where

$$e(\theta) = \begin{pmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ -\sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix}$$

and  $a(\rho)$  and  $\epsilon(\alpha)$  are given by eqn (30) and (33) respectively. We therefore obtain the following parametrization of the hyperbolic elements of  $g \in \text{SL}(2, \mathbb{R})$ ,

$$g = e(\theta)a(\rho)\epsilon(\sigma)a^{-1}(\rho)e^{-1}(\theta)$$

The use of the isomorphism kernel in equation (25) then yields

$$\alpha = \cosh \frac{\sigma}{2} + i \sinh \frac{\sigma}{2} \sinh \rho \tag{34}$$

$$\beta = -ie^{-i\theta} \sinh \frac{\sigma}{2} \cosh \rho \tag{35}$$

In Bargmann's theory<sup>7</sup> the carrier space for the discrete series of representations of  $\text{SU}(1,1)$  was taken to be the functions  $\phi(z_1, z_2)$  where  $z_1, z_2$  are the spinors transforming according to the fundamental representation of  $\text{SU}(1,1)$ :

$$(z'_1, z'_2) = (z_1, z_2)u$$

Since the fundamental representation of  $\text{SU}(1,1)$  and its complex conjugate are equivalent the functions  $\phi(z_1, z_2)$  are required to satisfy

$$\frac{\partial}{\partial \bar{z}_1} \phi(z_1, z_2) = \frac{\partial}{\partial \bar{z}_2} \phi(z_1, z_2) = 0,$$

so that  $\phi(z_1, z_2)$  is an analytic function of  $z_1$  and  $z_2$ . The generators of  $SU(1,1)$  in this realization are given by,

$$\begin{aligned} J_1 &= \frac{i}{2} \left( z_1 \frac{\partial}{\partial z_2} + z_2 \frac{\partial}{\partial z_1} \right) \\ J_2 &= \frac{1}{2} \left( z_1 \frac{\partial}{\partial z_2} - z_2 \frac{\partial}{\partial z_1} \right) \\ J_3 &= \frac{1}{2} \left( z_1 \frac{\partial}{\partial z_1} - z_2 \frac{\partial}{\partial z_2} \right) \end{aligned}$$

where  $J_3$  is the space rotation, and,  $J_1$  and  $J_2$  are pure Lorentz boosts.

Explicit calculation yields,

$$J_1^2 + J_2^2 - J_3^2 = K(1 - K)$$

where,

$$K = -\frac{1}{2} \left( z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} \right)$$

Since  $K$  commutes with  $J_1$ ,  $J_2$  and  $J_3$  it follows that in an irreducible representation  $\phi(z_1, z_2)$  is a homogeneous function of degree  $-2k$ ,

$$\phi(z_1, z_2) = z_2^{-2k} f(z), \quad z = \frac{z_1}{z_2},$$

where  $f(z)$  is an analytic function of  $z$ . It should be pointed out that under the action of  $SU(1,1)$  the complex  $z$ -plane is foliated into three orbits a)  $|z| < 1$ , b)  $|z| > 1$  and c)  $|z| = 1$  and the positive discrete series  $D_k^+$  ( $k = \frac{1}{2}, 1, \frac{3}{2}, \dots$ ) is described by the first orbit  $|z| < 1$ , the open unit disc. Thus in Bargmann's construction the subspace  $B_k(C)$  for  $D_k^+$  consists of functions  $f(z)$  analytic within the unit disc.

The finite element of the group in this realization can be easily obtained and is given by

$$T_u f(z) = (\beta z + \bar{\alpha})^{-2k} f\left(\frac{\alpha z + \bar{\beta}}{\beta z + \bar{\alpha}}\right) \quad (36)$$

These representations are unitary under the scalar product

$$(f, g) = \int_{|z|<1} \overline{f(z)} g(z) d\lambda(z) \quad (37)$$

where

$$d\lambda(z) = \frac{(2k-1)}{\pi} (1-|z|^2)^{2k-2} d^2 z, \quad (38)$$

$$z = x + iy, d^2 z = dx dy$$

The principal vector in  $B_k(C)$  is given by

$$e_z(z_1) = (1 - \bar{z} z_1)^{-2k} \quad (39)$$

so that

$$f(z) = \int_{|z_1|<1} (1 - z \bar{z}_1)^{-2k} f(z_1) d\lambda(z_1) \quad (40)$$

The action of the group ring,

$$T_x = \int x(u) T_u d\mu(u)$$

where  $d\mu(u)$  is the invariant measure on  $SU(1,1)$  is given by

$$T_x f(z) = \int x(u) (\beta z + \bar{\alpha})^{-2k} f\left(\frac{\alpha z + \bar{\beta}}{\beta z + \bar{\alpha}}\right) d\mu(u) \quad (41)$$

Now, as before, using the basic property of the principal vector *i.e.* ( 40) we have

$$f\left(\frac{\alpha z + \bar{\beta}}{\beta z + \bar{\alpha}}\right) = \int_{|z_1|<1} \left[1 - \frac{(\alpha z + \bar{\beta}) \bar{z}_1}{(\beta z + \bar{\alpha})}\right]^{-2k} f(z_1) d\lambda(z_1) \quad (42)$$

Substituting Eq. ( 42) in Eq. ( 41) we immediately obtain,

$$T_x f(z) = \int_{|z_1|<1} K(z, z_1) f(z_1) d\lambda(z_1)$$



where

$$K(z, z_1) = \int x(u)(\beta z + \bar{\alpha})^{-2k} \left[ 1 - \frac{(\alpha z + \bar{\beta})\bar{z}_1}{(\beta z + \bar{\alpha})} \right]^{-2k} d\mu(u) \quad (43)$$

Since the kernel, once again, is of the Hilbert-Schmidt type we have

$$Tr(T_x) = \int_{|z|<1} K(z, z) d\lambda(z)$$

Using Eq. (43), the definition of the integral kernel, the above equation can be written in the form

$$Tr(T_x) = \int x(u)\pi(u)d\mu(u)$$

where the character  $\pi(u)$  is given by

$$\pi(u) = \int_{|z|<1} (\beta z + \bar{\alpha})^{-2k} \left[ 1 - \frac{(\alpha z + \bar{\beta})\bar{z}}{(\beta z + \bar{\alpha})} \right]^{-2k} d\lambda(z) \quad (44)$$

We first consider the above integral for the elliptic elements of  $SU(1,1)$ . Setting

$$z = \tanh \frac{\tau}{2} e^{i\theta}, \quad 0 \leq \tau < \infty, \quad 0 \leq \theta \leq 2\pi \quad (45)$$

and using the parametrization (31,32) for the elliptic elements we obtain after some calculations

$$\pi(u) = \frac{2k-1}{4\pi} \int_{\tau=0}^{\infty} d\tau \sinh \tau \int_0^{2\pi} d\phi \left[ \cos \frac{\theta_0}{2} - i \sin \frac{\theta_0}{2} (\cosh \sigma \cosh \tau + \sinh \sigma \sinh \tau \cos \phi) \right]^{-2k} \quad (46)$$

where  $\phi = \eta + \theta$ . We now introduce the time-like  $SO(2,1)$  unit vectors,

$$\hat{n} = (-\sinh \sigma, 0, \cosh \sigma), \hat{r} = (\sinh \tau \cos \phi, \sinh \tau \sin \phi, \cosh \tau)$$

. Then the Eq. (46) can be written as

$$\pi(u) = \frac{2k-1}{4\pi} \int_{\tau=0}^{\infty} d\tau \sinh \tau \int_0^{2\pi} d\phi \left[ \cos \frac{\theta_0}{2} - i \sin \frac{\theta_0}{2} \hat{n} \cdot \hat{r} \right]^{-2k} \quad (47)$$

where  $\hat{n}.\hat{r}$  stands for the Lorentz invariant form

$$\hat{n}.\hat{r} = \hat{n}_3\hat{r}_3 - \hat{n}_2\hat{r}_2 - \hat{n}_1\hat{r}_1$$

Let us now perform a Lorentz transformation such that the time axis coincides with the fixed time-like SO(2,1) vector  $\hat{n}$ . Thus

$$\hat{n}.\hat{r} = \cosh \tau,$$

and we have

$$\pi(u) = \frac{2k-1}{2} \int_{\tau=0}^{\infty} \left[ \cos \frac{\theta_0}{2} - i \sin \frac{\theta_0}{2} \cosh \tau \right]^{-2k} \sinh \tau d\tau$$

The above integration is quite elementary and it leads to Gel'fand and coworkers' formula for the character of the elliptic elements of SU(1,1)

$$\pi(u) = \frac{e^{\frac{i}{2}\theta_0(2k-1)}}{e^{-i\frac{\theta_0}{2}} - e^{i\frac{\theta_0}{2}}}$$

For the hyperbolic elements of SU(1,1) we substitute the parametrization (34,35) in Eq. (44) and use the transformation (45). Thus

$$\pi(u) = \frac{2k-1}{4\pi} \int_{\tau=0}^{\infty} d\tau \sinh \tau \int_0^{2\pi} d\theta \left[ \cosh \frac{\sigma}{2} - i\hat{n}.\hat{r} \sinh \frac{\sigma}{2} \right]^{-2k} \quad (48)$$

where

$$\hat{n} = (-\cosh \rho \cos \eta, -\cosh \rho \sin \eta, \sinh \rho)$$

is a fixed space-like unit SO(2,1) vector and

$$\hat{r} = (\sinh \tau \cos \theta, \sinh \tau \sin \theta, \cosh \tau)$$

is a time-like unit SO(2,1) vector. If we now perform a Lorentz transformation such that the first space axis (X-axis) coincides with the fixed space-like SO(2,1) vector  $\hat{n}$  then

$$\hat{n}.\hat{r} = \sinh \tau \cos \theta$$

so that

$$\pi(u) = \frac{2k-1}{4\pi} \int_{\tau=0}^{\infty} d\tau \sinh \tau \int_{\theta=0}^{2\pi} d\theta \left[ \cosh \frac{\sigma}{2} - i \sinh \tau \cos \theta \sinh \frac{\sigma}{2} \right]^{-2k} \quad (49)$$

The evaluation of this integral is a little lengthy and is relegated to the appendix. Its value is given by

$$\pi(u) = \frac{e^{-\frac{1}{2}\sigma(2k-1)}}{e^{\frac{\sigma}{2}} - e^{-\frac{\sigma}{2}}}$$

### III. THE PRINCIPAL SERIES OF REPRESENTATIONS

For the representations of the principal series we shall realize the representations in the Hilbert space of functions<sup>6</sup> defined on the unit circle. For the representations of the integral class

$$T_u f(e^{i\theta}) = |\beta e^{i\theta} + \bar{\alpha}|^{-2k} f\left(\frac{\alpha e^{i\theta} + \bar{\beta}}{\beta e^{i\theta} + \bar{\alpha}}\right)$$

For the representations of the half-integral class,

$$T_u f(e^{i\theta}) = |\beta e^{i\theta} + \bar{\alpha}|^{-2k-1} (\beta e^{i\theta} + \bar{\alpha}) f\left(\frac{\alpha e^{i\theta} + \bar{\beta}}{\beta e^{i\theta} + \bar{\alpha}}\right)$$

In both the cases

$$k = \frac{1}{2} - is \quad , \quad -\infty < s < \infty$$

In what follows we shall consider the integral class first. For later convenience we replace  $e^{i\theta}$  by  $\exp[i(\theta - \pi/2)] = -ie^{i\theta}$ . Thus

$$T_u f(-ie^{i\theta}) = |-i\beta e^{i\theta} + \bar{\alpha}|^{-2k} f\left(\frac{-i\alpha e^{i\theta} + \bar{\beta}}{-i\beta e^{i\theta} + \bar{\alpha}}\right)$$

We now construct the group ring

$$T_x = \int x(u) T_u d\mu(u)$$

so that

$$T_x f(-ie^{i\theta}) = \int x(u) | -i\beta e^{i\theta} + \bar{\alpha} |^{-2k} f\left(\frac{-i\alpha e^{i\theta} + \bar{\beta}}{-i\beta e^{i\theta} + \bar{\alpha}}\right) d\mu(u)$$

We now make a left translation

$$u \rightarrow \underline{\theta}^{-1} u$$

where

$$\underline{\theta} = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$$

so that

$$\alpha \rightarrow \alpha e^{-i\theta/2} \quad , \quad \beta \rightarrow \beta e^{-i\theta/2}$$

We therefore obtain

$$T_x f(-ie^{i\theta}) = \int x(\theta^{-1}u) | -i\beta + \bar{\alpha} |^{-2k} f\left(\frac{-i\alpha + \bar{\beta}}{-i\beta + \bar{\alpha}}\right) d\mu(u)$$

We now map the  $SU(1,1)$  matrix  $u$  onto the  $SL(2, R)$  matrix  $g$  by using the isomorphism kernel  $\eta$  given by eqns.(25) and (26) and perform the Iwasawa decomposition

$$g = k \theta_1 \tag{50}$$

where

$$k = \begin{pmatrix} k_{11} & k_{12} \\ 0 & k_{22} \end{pmatrix}, \quad k_{11}k_{22} = 1 \tag{51}$$

belongs to the subgroup  $K$  of real triangular matrices of determinant unity and  $\theta_1 \in \Theta$  where  $\Theta$  is the subgroup of pure rotation matrices:

$$\theta_1 = \begin{pmatrix} \cos(\theta_1/2) & -\sin(\theta_1/2) \\ \sin(\theta_1/2) & \cos(\theta_1/2) \end{pmatrix} \quad (52)$$

We now introduce the following convention. The letters without a bar below it will indicate the  $SL(2, R)$  matrices or its subgroups and those with a bar below it will indicate their  $SU(1, 1)$  image. For instance

$$\underline{k} = \eta \ k \ \eta^{-1} = \frac{1}{2} \begin{pmatrix} k_{11} + k_{22} - ik_{12} & k_{12} - i(k_{11} - k_{22}) \\ k_{12} + i(k_{11} - k_{22}) & k_{11} + k_{22} + ik_{12} \end{pmatrix}$$

$$\underline{\theta}_1 = \begin{pmatrix} e^{i\theta_1/2} & 0 \\ 0 & e^{-i\theta_1/2} \end{pmatrix}$$

The decomposition (50) can also be written as

$$u = \underline{k} \ \underline{\theta}_1 \quad (53)$$

which yields

$$-i \ \alpha \ + \ \bar{\beta} \ = \ -i \ k_{22} \ e^{i\theta_1/2}$$

$$-i \ \beta \ + \ \bar{\alpha} \ = \ k_{22} \ e^{-i\theta_1/2}$$

Hence setting  $f(-ie^{i\theta}) = g(\theta)$  we obtain,

$$T_x \ g(\theta) = \int x(\underline{\theta}^{-1} \ \underline{k} \ \underline{\theta}_1) \ | \ k_{22} \ |^{-2k} \ g(\theta_1) \ d\mu(u) \quad (54)$$

It can be shown that under the decomposition (50,51,52) or equivalently (53) the invariant measure decomposes as

$$d\mu(u) = \frac{1}{2} d\mu_l(g) = \frac{1}{2} d\mu_r(g) = \frac{1}{4} d\mu_l(k) d\theta_1 \quad (55)$$

Substituting the decomposition (55) in eq. (54) we have

$$T_x g(\theta) = \int_{\Theta} K(\theta, \theta_1) g(\theta_1) d\theta_1$$

where

$$K(\theta, \theta_1) = \frac{1}{4} \int_K x(\underline{\theta}^{-1} \underline{k} \underline{\theta}_1) |k_{22}|^{-2k} d\mu_l(k) \quad (56)$$

Since the kernel  $K(\theta, \theta_1)$  is of the Hilbert-Schmidt type it has the trace

$$Tr(T_x) = \int_{\Theta} K(\theta, \theta) d\theta$$

Using the definition of the kernel as given by eq.(56) we have

$$Tr(T_x) = \frac{1}{4} \int_{\Theta, K} x(\underline{\theta}^{-1} \underline{k} \underline{\theta}) |k_{22}|^{-2k} d\theta d\mu_l(k) \quad (57)$$

Before proceeding any farther we note that  $\underline{\theta}^{-1} \underline{k} \underline{\theta}$  represents a hyperbolic element of  $SU(1, 1)$ :

$$u = \underline{\theta}^{-1} \underline{k} \underline{\theta}$$

Calculating the trace of both sides we have

$$k_{22} + 1/k_{22} = 2\alpha_1 \quad (58)$$

In the previous section we have seen that for the elliptic elements of  $SU(1, 1)$

$\alpha_1 = \cos(\theta_0/2) < 1$ . The eqn.(58), therefore, for the elliptic case yields

$$k_{22}^2 - 2k_{22} \cos(\theta_0/2) + 1 = 0$$

which has no real solution. Thus the elliptic elements of  $SU(1, 1)$  do not contribute to the character of the principal series of representations. We, therefore, assert that for this particular class of unirreps the trace is concentrated on the hyperbolic elements.

We shall now show that every hyperbolic element of  $SU(1, 1)$  ( i.e those with  $|\alpha_1| = |(a + d)|/2 > 1$  ) can be represented as

$$u = \underline{\theta}^{-1} \underline{k} \underline{\theta} \quad (59)$$

or equivalently as

$$g = \theta^{-1} k \theta \quad (60)$$

Here  $k_{11} = \lambda^{-1}$ ,  $k_{22} = \lambda$  are the eigenvalues of the matrix  $g$  taken in any order.

We recall that every  $g \in SL(2, R)$  for the hyperbolic case can be diagonalized as

$$v' g v'^{-1} = \delta$$

where

$$\delta = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}, \quad \delta_1 \delta_2 = 1; \quad \delta_1, \delta_2 \text{ real}$$

belongs to the subgroup  $D$  of real diagonal matrices of determinant unity and

$v' \in SL(2, R)$ . If we write the Iwasawa decomposition for  $v'$

$$v' = k' \theta$$

then

$$g = \theta^{-1} k'^{-1} \delta k' \theta$$

Now  $k'^{-1} \delta k' \in K$  so that writing  $k = k'^{-1} \delta k'$  we have the decomposition (60) in which

$$k_{11} = \delta_1 = \lambda^{-1}, \quad k_{22} = \delta_2 = \lambda$$

If these eigenvalues are distinct then for a given ordering of them the matrices  $k$ ,  $\theta$  are determined uniquely by the matrix  $g$ . In fact we have

$$k_{12} = b - c, \quad \tan \theta = \frac{(a - d)(b - c) + (\lambda - \lambda^{-1})(b + c)}{(b - c)(b + c) + (\lambda - \lambda^{-1})(a - d)}$$

It follows that for a given choice of  $\lambda$  the parameters  $\theta$  and  $k_{12}$  are uniquely determined. We note that there are exactly two representations of the matrix  $g$  by means of formula (60) corresponding to two distinct possibilities

$$k_{11} = \operatorname{sgn} \lambda |\lambda|^{-1} = \operatorname{sgn} \lambda e^{\sigma/2}, \quad k_{22} = \operatorname{sgn} \lambda |\lambda| = \operatorname{sgn} \lambda e^{-\sigma/2}$$

$$k_{11} = \operatorname{sgn} \lambda |\lambda| = \operatorname{sgn} \lambda e^{-\sigma/2}, \quad k_{22} = \operatorname{sgn} \lambda |\lambda|^{-1} = \operatorname{sgn} \lambda e^{\sigma/2}$$

Let us now remove from  $K$  the elements with  $k_{11} = k_{22} = 1$ . This operation cuts the group  $K$  into two connected disjoint components. Neither of these components contains two matrices which differ only by permutation of the two diagonal elements. In correspondence with this partition the integral in the r.h.s. of eqn.(57) is represented in the form of a sum of two integrals

$$\operatorname{Tr} (T_x) = \frac{1}{4} \int_{\Theta} d\theta \int_{K_1} d\mu_l(k) |k_{22}|^{-2k} x(\underline{\theta}^{-1} \underline{k} \underline{\theta}) + \frac{1}{4} \int_{\Theta} d\theta \int_{K_2} d\mu_l(k) |k_{22}|^{-2k} x(\underline{\theta}^{-1} \underline{k} \underline{\theta}) \quad (61)$$

As  $\theta$  runs over the subgroup  $\Theta$  and  $k$  runs over the components  $K_1$  or  $K_2$  the matrix  $g = \theta^{-1} k \theta$  runs over the hyperbolic elements of the group  $SL(2, R)$  or equivalently  $u = \underline{\theta}^{-1} \underline{k} \underline{\theta}$  runs over the hyperbolic elements of the  $SU(1, 1)$ . We shall now prove that in  $K_1$  or  $K_2$

$$d\mu_l(k) d\theta = \frac{4 |k_{22}|}{|k_{22} - k_{11}|} d\mu(u) \quad (62)$$

To prove this we start from the left invariant differential element

$$dw = g^{-1} dg$$



where  $g \in SL(2, R)$  and  $dg$  is the matrix of the differentials  $g_{pq}$  i.e. following the notation of eqn.(24)

$$dg = \begin{pmatrix} da & db \\ dc & dd \end{pmatrix}$$

The elements  $dw$  are invariant under the left translation  $g \rightarrow g_0 g$ . Hence choosing a basis in the set of all  $dg$  we immediately obtain a differential left invariant measure. For instance choosing  $dw_{12}$ ,  $dw_{21}$ ,  $dw_{22}$  as the independent invariant differentials we arrive at the left invariant measure on  $SL(2, R)$ .

$$d\mu_l(g) = dw_{12}dw_{21}dw_{22} \quad (63)$$

In a similar fashion we can define the right invariant differentials

$$dw' = dg \, g^{-1}$$

which is invariant under the right translation  $g \rightarrow gg_0$ .

To prove the formula (62) we write the decomposition (60) as

$$\theta \, g = k \, \theta$$

so that

$$d\theta \, g + \theta \, dg = dk \, \theta + k \, d\theta \quad (64)$$

From eqn.(64) it easily follows that

$$dw = g^{-1} \, dg = \theta^{-1} \, d\mu \, \theta \quad (65)$$

where

$$d\mu = k^{-1} \, dk + d\theta \, \theta^{-1} - k^{-1} \, d\theta \, \theta^{-1} \, k \quad (66)$$

In accordance with the choice of the independent elements of  $dw$  as mentioned above we choose the independent elements of  $d\mu$  as  $d\mu_{12}$ ,  $d\mu_{21}$ ,  $d\mu_{22}$ . The eqn.(65) then leads to

$$dw_{11} + dw_{22} = d\mu_{11} + d\mu_{22} \quad (67)$$

$$dw_{11}dw_{22} - dw_{12}dw_{21} = d\mu_{11}d\mu_{22} - d\mu_{12}d\mu_{21} \quad (68)$$

Further since  $Tr(d\mu) = Tr(dw) = 0$  we immediately obtain from eqn.(67,68)

$$dw_{22}^2 + dw_{12}dw_{21} = d\mu_{22}^2 + d\mu_{12}d\mu_{21}$$

which can be written in the form

$$d\eta_1^2 + d\eta_2^2 - d\eta_3^2 = d\eta_1'^2 + d\eta_2'^2 - d\eta_3'^2 \quad (69)$$

where

$$\begin{aligned} d\eta_1 &= (dw_{12} + dw_{21})/2, & d\eta_1' &= (d\mu_{12} + d\mu_{21})/2 \\ d\eta_2 &= dw_{22}, & d\eta_2' &= d\mu_{22} \\ d\eta_3 &= (dw_{12} - dw_{21})/2, & d\eta_3' &= (d\mu_{12} - d\mu_{21})/2 \end{aligned}$$

The eqn.(69) implies that the set  $d\eta$  and the set  $d\eta'$  are connected by a Lorentz transformation. Since the volume element  $d\eta_1 d\eta_2 d\eta_3$  is invariant under such a transformation we have,

$$d\eta_1 d\eta_2 d\eta_3 = d\eta_1' d\eta_2' d\eta_3' \quad (70)$$

But the l.h.s. of eqn.(70) is  $dw_{12}dw_{21}dw_{22}/2$  and r.h.s. is  $d\mu_{12}d\mu_{21}d\mu_{22}/2$ . Hence using eqn.(63) we easily obtain

$$d\mu_i(g) = d\mu_{12}d\mu_{21}d\mu_{22}$$

We now write eqn.(66) in the form

$$d\mu = du + dv \quad (71)$$

where  $du = k^{-1}dk$  is the left invariant differential element on  $K$  and

$$dv = d\theta \theta^{-1} - k^{-1} d\theta \theta^{-1} k \quad (72)$$

In eqn.(71)  $du$  is a triangular matrix whose independent nonvanishing elements are chosen to be  $du_{12}$ ,  $du_{22}$  so that

$$d\mu_l(k) = du_{12}du_{22}$$

On the other hand  $dv$  is a  $2 \times 2$  matrix having one independent element which is chosen to be  $dv_{21}$ . Since the Jacobian connecting  $d\mu_{12}d\mu_{21}d\mu_{22}$  and  $du_{12}du_{22}dv_{21}$  is a triangular determinant having 1 along the main diagonal we obtain

$$d\mu_l(g) = d\mu_l(k) dv_{21} \quad (73)$$

It can now be easily verified that each element  $k \in K$  with distinct diagonal elements (which is indeed the case for  $K_1$  or  $K_2$ ) can be represented uniquely in the form

$$k = \zeta^{-1} \delta \zeta \quad (74)$$

where  $\delta$  belongs to the subgroup of real diagonal matrices with unit determinant and  $\zeta \in Z$ , where  $Z$  is a subgroup of  $K$  consisting of real matrices of the form

$$\zeta = \begin{pmatrix} 1 & \zeta_{12} \\ 0 & 1 \end{pmatrix}$$

Writing eqn.(74) in the form  $\zeta k = \delta \zeta$  we obtain

$$k_{pp} = \delta_p \quad , \quad \zeta_{12} = k_{12}/(\delta_1 - \delta_2) \quad (75)$$

Using the decomposition (74) we can now write eqn.(72) in the form

$$dv = \zeta^{-1} dp \zeta$$

where

$$dp = d\lambda - \delta^{-1} d\lambda \delta$$

$$d\lambda = \zeta d\sigma \zeta^{-1} \quad , \quad d\sigma = d\theta \theta^{-1}$$

From the above equations it now easily follows

$$dv_{21} = \frac{|\delta_2 - \delta_1|}{|\delta_2|} \frac{d\theta}{2} \quad (76)$$

Substituting eqn.(76) in eqn.(73) and using eqns.(55) and (75) we immediately obtain eqn.(62).

Now recalling that in  $K_1$ ,  $|k_{22}| = e^{-\sigma/2}$  and in  $K_2$ ,  $|k_{22}| = e^{\sigma/2}$  the eqn.(61) in conjunction with (62) yields

$$Tr (T_x) = \int x(u) \pi(u) d\mu(u)$$

where the character  $\pi(u)$  is given by

$$\pi(u) = \frac{e^{(2k-1)\sigma/2} + e^{-(2k-1)\sigma/2}}{|e^{\sigma/2} - e^{-\sigma/2}|}$$

For the principal series of representations belonging to the half-integral class a parallel calculation yields

$$\pi(u) = \frac{e^{(2k-1)\sigma/2} + e^{-(2k-1)\sigma/2}}{|e^{\sigma/2} - e^{-\sigma/2}|} \operatorname{sgn} \lambda$$

## APPENDIX A: EVALUATION OF THE INTEGRAL ( 49)

By a simple change of variable the eq.(49) can be written in the form,

$$\pi(u) = \left[ \frac{(2k-1)}{2\pi} \right] \int_0^\infty \sinh \tau d\tau \int_0^\pi d\theta \left[ \cosh \left( \frac{\sigma}{2} \right) - i \sinh \left( \frac{\sigma}{2} \right) \cos \theta \sinh \tau \right]^{-2k}$$

To carry out the  $\theta$  integration we set

$$x = \frac{1}{2}(1 - \cos \theta), \quad 0 \leq x \leq 1$$

and use the formula<sup>8</sup>

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 x^{b-1} (1-x)^{c-b-1} (1-zx)^{-a} dx, \quad \text{Re}(c) > \text{Re}(b) > 0$$

so that

$$\pi(u) = \left[ \frac{(2k-1)}{2} \right] \int_0^\infty \sinh \tau d\tau \left[ \cosh \left( \frac{\sigma}{2} \right) - i \sinh \left( \frac{\sigma}{2} \right) \sinh \tau \right]^{-2k} F \left( 2k, \frac{1}{2}, 1; \frac{-2i \sinh \frac{\sigma}{2} \sinh \tau}{\left[ \cosh \left( \frac{\sigma}{2} \right) - i \sinh \frac{\sigma}{2} \sinh \tau \right]} \right)$$

To evaluate the above integral we use the quadratic transformation<sup>9</sup>

$$F(a, b; 2b; z) = \left( 1 - \frac{z}{2} \right)^{-a} F \left( \frac{a}{2}, \frac{1}{2} + \frac{a}{2}; b + \frac{1}{2}; \left[ \frac{z}{2-z} \right]^2 \right)$$

Thus

$$\pi(u) = \left[ \frac{(2k-1)}{2} \right] \int \left[ \cosh \left( \frac{\sigma}{2} \right) \right]^{-2k} F \left( k, \frac{1}{2} + k; 1; -\tanh^2 \left( \frac{\sigma}{2} \right) \sinh^2 \tau \right) \sinh \tau d\tau$$

For integral  $k$  we now extract the branch point of the hypergeometric function by using<sup>10</sup>

$$F(a, b; c; z) = (1-z)^{-b} F(c-a, b; c; \frac{z}{z-1})$$

Using the above formula we obtain after some calculations

$$\pi(u) = \left[ \frac{(2k-1)}{2} \right] \left[ \cosh \left( \frac{\sigma}{2} \right) \right]^{-2k} \left[ \text{sech}^2 \left( \frac{\sigma}{2} \right) \right]^{-\frac{1}{2}-k} \sum_{n=0}^{k-1} \frac{(1-k)_n (\frac{1}{2} + k)_n}{(1)_n n!} \left[ \sinh^2 \left( \frac{\sigma}{2} \right) \right]^n \int_0^\infty \left[ 1 + \sinh^2 \left( \frac{\sigma}{2} \right) \cosh^2 \tau \right]^{-\frac{1}{2}-k-n} (\cosh^2 \tau - 1)^n \sinh \tau d\tau$$

Setting  $\cosh \tau = t^{-\frac{1}{2}}$  we obtain after some calculations

$$\pi(u) = \left[ \frac{(2k-1)}{4} \right] \left[ \cosh \left( \frac{\sigma}{2} \right) \right] \left[ \sinh^2 \left( \frac{\sigma}{2} \right) \right]^{-\frac{1}{2}-k} \Gamma(k) \sum_{n=0}^{k-1} \frac{(1-k)_n (\frac{1}{2}+k)_n}{n! \Gamma(n+k+1)} F \left( \frac{1}{2} + k + n, k; n + k + 1; -\operatorname{cosech}^2 \left( \frac{\sigma}{2} \right) \right) \quad (\text{A1})$$

The summation over  $n$  can be carried out by expanding the hypergeometric function appearing in Eq. (A1) in a power series. Thus,

$$\pi(u) = \left[ \frac{(2k-1)}{4} \right] \left[ \cosh \left( \frac{\sigma}{2} \right) \right] \left[ \sinh^2 \left( \frac{\sigma}{2} \right) \right]^{-\frac{1}{2}-k} \frac{\Gamma(k)}{\Gamma(\frac{1}{2}+k)} \sum_{n=0}^{\infty} \frac{(k)_n (\frac{1}{2}+k+n)}{n! \Gamma(n+k+1)} \left[ -\operatorname{cosech}^2 \left( \frac{\sigma}{2} \right) \right]^n F \left( 1-k, \frac{1}{2} + k + n; n + k + 1; 1 \right)$$

The hypergeometric function of unit argument can be summed by Gauss's formula<sup>11</sup> and we obtain after some calculations

$$\pi(u) = \cosh \left( \frac{\sigma}{2} \right) \left[ \sinh^2 \left( \frac{\sigma}{2} \right) \right]^{-\frac{1}{2}-k} (2)^{-2k} F \left( k, k + \frac{1}{2}; 2k; -\operatorname{cosech}^2 \left( \frac{\sigma}{2} \right) \right)$$

We now use the formula<sup>12</sup>

$$F \left( k, k + \frac{1}{2}; 2k; z \right) = (1-z)^{-\frac{1}{2}} \left[ \frac{1}{2} (1 + \sqrt{1-z}) \right]^{(1-2k)}$$

which immediately yields

$$\pi(u) = \frac{e^{-\frac{1}{2}\sigma(2k-1)}}{e^{\frac{\sigma}{2}} - e^{-\frac{\sigma}{2}}}$$

for half-integral  $k$  a parallel calculation yields the same result.

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- <sup>11</sup> See ref. 8, p.104.
- <sup>12</sup> See ref. 8, p.101.